Distribution of droplets in a turbulent spray

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A statistical description of droplets in turbulent spray, connected with the description of turbulent dissipation, is presented. The ideas of similarity, the cascade processes, and the infinitely divisible distributions are used in this description. Formulas for characteristic droplet sizes and corresponding probability distributions are obtained along with a simple formula for turbulent dissipation in flow near a ship.
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I. INTRODUCTION

Turbulent spray is an essential part of a beautiful show, produced, particularly, by a moving ship. Apart from the beauty, the ship spray has some additional properties: it obscures vision, in cold weather it leads to the icing of a ship, and it produces a characteristic radar and visual signature in the shape of a ''necklace.'' Turbulent spray has many technical applications in combustion, painting, printing, etc.

The physics of turbulent spray is very complex. The problem of turbulence itself is far from being ''solved'' in any sense. The nonlinear boundary condition on the free surface introduces an additional complexity, connected with the nonlinear surface waves and their interaction with turbulence. On top of this we have the difficult problems of the droplet formation and separation. Direct numerical simulation of turbulent spray is beyond the reach of current computer capabilities. However, numerical modeling of turbulent spray with the use of the large-eddy simulation is in progress and the results will be presented elsewhere. Here we will describe a much more simple statistical approach, which is complementary to the numerical simulations.

Kolmogorov [1] in cooperation with the experimental group of Baranaev *et al.* [2] (see also a presentation of these works in Levich [3]) introduced a similarity argument to describe the characteristic size of droplets forming from a submerged jet of insoluble fluid in a turbulent flow of water. We will extend this type of argument to include recent results on turbulence intermittency. We will also obtain the probability distribution of the droplet sizes in turbulent spray.

II. DROPLET SIZES

Consider a well-developed turbulent flow near a water-air interface. The major parameters for turbulent spray sheets and jets are the mean rate of the turbulent energy dissipation in water $\langle\varepsilon\rangle$, the kinematic coefficient of surface tension $\gamma=\tau/\rho$ (τ is the dynamical surface tension, ρ is the density of water), the kinematic coefficient of viscosity of water ν, and the external scale of turbulence in water L. Additionally, we have the ratio between densities of air and water ρ_a/ρ, kinematic viscosity of air ν_a, and characteristics of turbulence in air: mean air velocity relative to water V_a, mean rate of turbulent energy dissipation $\langle\varepsilon_a\rangle$, and external scale of turbulence L_a. These additional parameters are important only in situations with strong wind or strong air flow, because the relative dynamical contribution of air to the spray formation, roughly speaking, is proportional to the small parameter ρ_a/ρ.

Neglecting the influence of the air flow, from a dimensional argument we obtain the following expression for the characteristic size of droplets in turbulent spray:

$$l_*=\gamma^{3/5}\langle\varepsilon\rangle^{-2/5}\Phi(m,W), \tag{1}$$

$$m=\nu^5\langle\varepsilon\rangle\gamma^{-4}, \quad W=\langle\varepsilon\rangle^{2/3}L^{5/3}\gamma^{-1}\sim v^2L\gamma^{-1}. \tag{2}$$

Here Φ is a nondimensional function of nondimensional arguments m (molecular viscosity effect) and the global Weber number W, v is the rms velocity fluctuation and in the definition of W we used the Kolmogorov's relation $v\sim\langle\varepsilon\rangle^{1/3}L^{1/3}$. Parameter m is typically very small and parameter W is large.

Consider the so-called inertial range of scales: $l_\nu\equiv\nu^{3/4}\langle\varepsilon\rangle^{-1/4}\ll l_*\ll L$, where l_ν is the Kolmogorov's internal scale. In this range, generally, we may expect that molecular viscosity ν and the external scale L are not important. In this case, from Eq. (1) we get

$$l_*\sim\gamma^{3/5}\langle\varepsilon\rangle^{-2/5}. \tag{3}$$

This relation signifies the energy (or pressure) balance between turbulence and surface tension for the droplet formation. Indeed, Eq. (3) can be written in the form

$$\langle u^2(r)\rangle\sim\langle\varepsilon\rangle^{2/3}r^{2/3}\sim\gamma r^{-1}, \quad r=l_*. \tag{4}$$

Here $u(r)$ is the velocity increment for distance r, angular brackets denote the statistical averaging and we used Kolmogorov's ''2/3 law.'' The term γr^{-1} represents the energy (pressure) due to surface tension. The spectral analog of Eq. (4) has the form

$$E(k)\sim\langle\varepsilon\rangle^{2/3}k^{-5/3}\sim\gamma, \quad k=l_*^{-1}, \tag{5}$$

where $E(k)$ is the energy spectrum, k is the wave number.

Relations (3)–(5) do not take into account the effects of intermittency. To consider these effects we can use the generalized Kolmogorov relation $u(r) \sim \varepsilon_r^{1/3} r^{1/3}$, where ε_r is the dissipation rate averaged over the distance r.

The balance equation (4) takes the form

$$\langle \varepsilon_r^{2/3} \rangle r^{2/3} \sim \gamma r^{-1}, \quad r = l_* . \tag{6}$$

Now we use the scale similarity of the breakdown coefficients $\varepsilon_r / \varepsilon_l$ [4–6] and the corresponding formula for the moment of order p:

$$\langle (\varepsilon_r / \varepsilon_l)^p \rangle = (l/r)^{\mu(p)}, \quad r \leqslant l. \tag{7}$$

Here $\mu(p)$ is the similarity exponent. With the usual assumption $\varepsilon_L \approx \langle \varepsilon \rangle$, from Eq. (7) we have

$$\langle \varepsilon_r^p \rangle \approx \langle \varepsilon \rangle^p (L/r)^{\mu(p)}. \tag{8}$$

Substitution of Eq. (8) with $p = 2/3$ into Eq. (6) gives

$$l_* \sim \gamma^{3/5} \langle \varepsilon \rangle^{-2/5} W^{\alpha}, \tag{9}$$

$$\alpha = -\frac{9\mu(2/3)}{25 - 15\mu(2/3)}, \tag{10}$$

where definition (2) for W was used. The same result follows from the spectral balance (5), if we use the corrected energy spectrum $E(k) \sim \varepsilon^{2/3} k^{-5/3} (kL)^{\mu(2/3)}$.

Thus, we get the intermittency correction for the characteristic size of droplets along with the dependence on the global Weber number. The quantity $\mu(2/3)$ is negative and small. Indeed, from Eq. (8) we see that $\mu(0) = 0$ (normalization of probability) and $\mu(1) = 0$ because $\langle \varepsilon_r \rangle = \langle \varepsilon \rangle$ (see details in [4–7], connected with the nonhomogeneity of the breakdown process). It was shown [5] that function $\mu(p)$ is concave. Thus, $\mu(2/3) < 0$. A general formula for $\mu(p)$, which satisfies the realizability and all other necessary and sufficient conditions, was obtained [6], based on the theory of the infinitely divisible distributions [see below formula (21)]. A particular case of this formula, which additionally satisfies the "no-gap" condition for the probability distribution [6], reads

$$\mu(p) = p - \frac{(ps+1)^{1/2} - 1}{(s+1)^{1/2} - 1} \quad (ps \geqslant -1). \tag{11}$$

Experimental data fit very well this formula with $s = 0.9126$ [8]. From Eqs. (11) and (10) we have $\mu(2/3) \approx -0.034$ and $\alpha \approx 0.012$. For turbulent flow around a battleship we may have $W \sim 10^6$ and the intermittency correction (9) gives about an 18% increase in the typical size of droplets in turbulent spray. The increase of l_* due to intermittency can be understood qualitatively without detailed calculations. In the energy (pressure) balance (6) the term $\langle \varepsilon_r^{2/3} \rangle$ is smaller than $\langle \varepsilon \rangle^{2/3}$, because in intermittent turbulent flow the areas with weak dissipation are larger than the areas with strong dissipation. Formula (8) with $\mu(2/3) < 0$ reflects this. Thus, l_* is determined by a weaker than average dissipation and we need a bigger l_* (a bigger droplet volume) to resist surface tension.

Experimental data were recently obtained for turbulent spray in a jet around a rod [9]. If we present these data in the form (9), then we get $\alpha \approx -0.14$. Previous consideration tells us that W dependence with such α cannot be explained by the intermittency effects. Our guess is that turbulent flow around a rod in these experiments was not in Kolmogorov's similarity regime. Assuming that the energy spectrum has the form $E(k) \sim \langle \varepsilon \rangle^{2/3} k^{-5/3} (kL)^{\beta}$, from the spectral balance (5), taking into account (2) and (9), we get

$$\alpha = -\frac{9\beta}{5(5 - 3\beta)}, \quad \beta = -\frac{25\alpha}{3(3 - 5\alpha)}. \tag{12}$$

If $\alpha = -0.14$, then from Eq. (12) we have $\beta \approx 0.315$, which means a significant deviation from the classical similarity regime. It seems very important in future experiments with turbulent spray to measure the energy spectrum of turbulence or equivalent structural function.

Returning to the ship spray, let us estimate the mean rate of the energy dissipation in turbulent flow in front of a moving ship. To begin with, imagine a vertical wall moving with speed V and creating turbulent waves. The characteristic velocity fluctuation of fluid will be of order of V and characteristic acceleration will be of order of the gravity acceleration g. By dimensional argument and by physical consideration we can estimate $\langle \varepsilon \rangle \sim Vg$. A ship is not a vertical wall and one of the goals of the ship design is to reduce turbulent dissipation. Introducing the corresponding form factor f_ε we have

$$\langle \varepsilon \rangle = Vg f_\varepsilon . \tag{13}$$

We did not find direct measurements of turbulent dissipation in bow waves, but by using Kolmogorov's formula $\langle \varepsilon \rangle \sim v^3 L^{-1}$ and data on wave measurements for different ship models [10], we can roughly estimate that f_ε is of order of 0.1. This factor enters Eq. (9) in the power $-2/5$, so uncertainty in this factor is not very consequential. For $V = 10$ m/s and $\gamma = 70$ cm^3/s^2 we get $l_* \sim 1$ mm. At the same time $l_\nu \sim 0.1$ mm, so l_* is in the inertial range but not far from the viscous cutoff. Formula (13) can be used not only for turbulent spray, but also for more general problems of the ship hydrodynamics.

III. PROBABILITY DISTRIBUTIONS

Now we turn to the probability distribution of droplet sizes, bearing in mind the close connection between droplet size and the energy dissipation, considered above. Particularly, we refer to the energy (pressure) balance (6) and to the increase of the characteristic droplet size l_* due to the intermittency of the dissipation. It seems natural to assume that distribution of droplets can be described similarly to the distribution of energy dissipation, although the individual breakdown coefficients may have different probability distributions. Let

$$l \equiv l_{N+1} = l_1 b_1 b_2 \cdots b_N , \quad b_k = l_{k+1}/l_k \leqslant 1. \tag{14}$$

Here l_1 is the size of a disintegrating blob of water, N is the number of stages of disintegration, l is the final size of an individual droplet. From Eq. (14) we have

$$y \equiv -\ln(l/l_1) = -\sum_{k=1}^{N} \ln b_k. \tag{15}$$

If we assume that coefficients b_k are statistically independent (or weakly dependent) and N is large, then bearing in mind the limit theorem, we may expect that the probability density distribution of the variable y will be normal:

$$P(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\{-(y-a)^2/2\sigma^2\}, \tag{16}$$

$$a = \langle y \rangle, \quad \sigma^2 = \langle (y-a)^2 \rangle. \tag{17}$$

However, it was shown [4] (in the context of the energy dissipation) that this conclusion is incorrect. Mathematically speaking, one has to be careful with the limit theorem: in our situation the properly normalized characteristic function of the probability distribution will tend to normal, but not the probability density function. Physically speaking, Eq. (16) contradicts the mass conservation. Indeed, simple calculation of moments by using Eq. (16) gives

$$\langle (l/l_1)^n \rangle = \exp\{-an + \sigma^2 n^2\}. \tag{18}$$

For large n, Eq. (18) exceeds unity, which contradicts the inequality in Eq. (14). It seems natural to assume that the proper probability distribution of droplets belongs to the family of the infinitely divisible distributions, suggested in [6], which satisfy all necessary and sufficient conditions. Here we present one example of such a distribution:

$$P(y) = \frac{a^{3/2}}{\sqrt{2\pi}\sigma y^{3/2}} \exp\left\{-\frac{a}{2\sigma^2}(ay^{-1/2} - y^{1/2})^2\right\}, \quad y \geqslant 0, \tag{19}$$

where a and σ are defined by Eq. (17). Moments of droplet size, corresponding to distribution (19), have the form

$$\langle (l/l_1)^n \rangle = \langle e^{-ny} \rangle = \exp\{-a^2\sigma^{-2}[(2\sigma^2 a^{-1} n + 1)^{1/2} - 1]\}. \tag{20}$$

This formula is analogous to Eq. (11), keeping in mind that $r\varepsilon_r/L\varepsilon_L$ is analogous to l/l_1 [see Ref. [6] and Problem 14, Chap. 13 in Ref. [11] for assistance in obtaining expression (20), see also formula (27) below].

More general distributions can be obtained from the following general formula [6]:

$$\mu(p) = \kappa p - \int_0^\infty \frac{1 - e^{-px}}{x} F(dx), \quad \kappa \leqslant 1. \tag{21}$$

Here κ is a constant, F is a measure on the open interval $(0,\infty)$ such that $(1+x)^{-1}$ is integrable with respect to F. An analytical expression for $\mu(p)$ can be obtained when the measure density has the following form:

$$F'(x) = \sum_i A_i \Gamma(\alpha_i)(x/s_i)^{\alpha_i - 1} \exp(-x/s_i) + \sum_j B_j x_j \delta(x - x_j). \tag{22}$$

Here $A_i, \alpha_i, s_i, B_j, x_j$ are, generally, positive constants [or such that the whole expression (22) is non-negative for $x \geqslant 0$, see example [7] with $A_2 = -A_1$, $s_1 > s_2$], Γ is the gamma function. Substitution of Eq. (22) onto Eq. (21) gives

$$\mu(p) = \kappa p - \sum_i A_i \left[\frac{(ps_i + 1)^{1-\alpha_i} - 1}{1 - \alpha_i}\right] - \sum_j B_j (1 - e^{-px_j}). \tag{23}$$

For $\alpha_i = 1$ the corresponding expression in square brackets in Eq. (23) should be replaced by $\ln(ps_i + 1)$.

Having $\mu(p)$, we can calculate the characteristic function for $\ln q_{r,l}$, where $q_{r,l} = \varepsilon_r/\varepsilon_l$ is the breakdown coefficient [see formula (7)]:

$$\psi(s, l/r) = \langle \exp(is \ln q_{r,l}) \rangle = (l/r)^{\mu(is)}. \tag{24}$$

The probability density function for $q_{r,l}$ is given by the expression [6]

$$W(q, l/r) = \frac{1}{2\pi q} \int_{-\infty}^{\infty} \exp[-is \ln q + \mu(is)\ln(l/r)] ds. \tag{25}$$

In the case of droplets, we can directly write expression for the characteristic function of the variable y, defined by Eq. (15) [6,11]:

$$\chi(s) \equiv \langle e^{isy} \rangle = \exp\left\{ibs - \int_0^\infty \frac{1 - e^{isx}}{x} Q(dx)\right\}, \quad b \geqslant 0. \tag{26}$$

Here b is a constant, Q is a measure with the same properties as measure F in Eq. (21), and we can use the same expression (22) for the measure density. The probability density for y is given by the Fourier transform of Eq. (26). Particularly, if we use only one first term in Eq. (22) with $\alpha_1 = 0.5$, we will get Eq. (19). The moments for the droplet size distribution we obtain from (26):

$$\langle (l/l_1)^n \rangle = \langle e^{-ny} \rangle = \chi(in) = \exp\left\{-bn - \int_0^\infty \frac{1 - e^{-nx}}{x} Q(dx)\right\}. \tag{27}$$

IV. CONCLUSION

The presented formulas for the similarity exponents and probability distributions describe both the intermitted dissipation and droplets in turbulent spray. The transformation from a statistical description of dissipation to droplets is illustrated. These results can be used in experimental and numerical studies of turbulent spray. A simple formula (13) for the turbulent dissipation for flow near a ship is proposed. We hope that this formula will stimulate direct measurements of turbulent dissipation for different ship models.

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